

ACCELERATED SPATIAL APPROXIMATIONS FOR TIME DISCRETIZED STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT. The present article investigates the convergence of a class of space-time discretization schemes for the Cauchy problem for linear parabolic stochastic partial differential equations (SPDEs) defined on the whole space. Sufficient conditions are given for accelerating the convergence of the scheme with respect to the spatial approximation to higher order accuracy by an application of Richardson's method. This work extends the results of Gyöngy and Krylov [*SIAM J. Math. Anal.*, 42 (2010), pp. 2275–2296] to schemes that discretize in time as well as space.

1. INTRODUCTION

For a fixed $\tau \in (0, 1)$, we consider the equation

$$(1.1) \quad v_i^h = v_{i-1}^h + (L_i^h v_i^h + f_i) \tau + \sum_{\rho=1}^{d_1} \left(M_{i-1}^{h,\rho} v_{i-1}^h + g_{i-1}^\rho \right) \xi_i^\rho$$

for $i \in \{1, \dots, n\}$ and $(\omega, x) \in \Omega \times G_h$ with a given initial condition, where G_h is the space grid

$$G_h := \{\lambda_1 h + \dots + \lambda_p h; \lambda_1, \dots, \lambda_p \in \Lambda \cup (-\Lambda)\}$$

with mesh size $h \in \mathbf{R} \setminus \{0\}$ for a finite subset $\Lambda \subset \mathbf{R}^d$, for integer $d \geq 1$, containing the origin. For a fixed $T \in (0, \infty)$ we define the time grid

$$T_\tau := \{t_i = i\tau; i \in \{0, 1, \dots, n\}, \tau n = T\},$$

partitioning $[0, T]$ with mesh size τ , and note that $v^h = v^h(\omega, t, x)$ depends on the parameter τ as well as h , since we have used the convention of writing v_i^h in place of $v^h(t_i)$ for $t_i \in T_\tau$. In particular, let $\xi_i^\rho = \Delta w^\rho(t_{i-1}) := w^\rho(t_i) - w^\rho(t_{i-1})$ be the i th increment of w^ρ with respect to T_τ , where, for integer $d_1 \geq 1$, $(w^\rho)_{\rho=1}^{d_1}$ is a given sequence of independent Wiener processes carried by the stochastic basis $(\Omega, \mathcal{F}, \mathcal{F}(t), P)$ that is complete with respect the filtration $\mathcal{F}(t)$ for $t \in [0, T]$. For each $i \in \{0, \dots, n\}$, the L_i^h and $M_i^{h,\rho}$ are difference operators given by $L_i^h \phi := \mathbf{a}_i^{\lambda\mu} \delta_{h,\lambda} \delta_{-h,\mu} \phi$ and $M_i^{h,\rho} \phi := \mathbf{b}_i^{\lambda\rho} \delta_{h,\lambda} \phi$,

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for $\rho \in \{1, \dots, d_1\}$, where repeated indices indicate summation over $\lambda, \mu \in \Lambda$. We assume that $\mathbf{a}_i^{\lambda\mu} = \mathbf{a}_i^{\lambda\mu}(x)$ and $\mathbf{b}_i^\lambda = (\mathbf{b}_i^{\lambda\rho}(x))_{\rho=1}^{d_1}$ are real-valued $\mathcal{P} \times \mathcal{B}$ -measurable functions on $\Omega \times T_\tau \times \mathbf{R}^d$ for all $\lambda, \mu \in \Lambda$ and further that $\mathbf{a}_i^{\lambda\mu} = \mathbf{a}_i^{\mu\lambda}$. Here \mathcal{P} denotes the σ -algebra of predictable subsets of $\Omega \times [0, \infty)$ generated by $\mathcal{F}(t)$ and $\mathcal{B} = \mathcal{B}(\mathbf{R}^d)$ denotes the σ -algebra of Borel subsets of \mathbf{R}^d . The spatial differences above are defined by

$$\delta_{h,\lambda}\phi(x) := \frac{\phi(x + h\lambda) - \phi(x)}{h}$$

for $\lambda \in \mathbf{R}^d \setminus \{0\}$ and by the identity for $\lambda = 0$. We note that from this definition one can obtain both the so called “forward” and “backward” differences as h can be positive or negative.

Together with (1.1) we consider

$$(1.2) \quad v_i = v_{i-1} + (\mathcal{L}_i v_i + f_i) \tau + \sum_{\rho=1}^{d_1} (\mathcal{M}_{i-1}^\rho v_{i-1} + g_{i-1}^\rho) \xi_i^\rho$$

for $i \in \{1, \dots, n\}$ and $(\omega, x) \in \Omega \times \mathbf{R}^d$ with a given initial condition. Here $\mathcal{L}_i = \mathcal{L}(t_i)$ and $\mathcal{M}_i^\rho = \mathcal{M}^\rho(t_i)$ are second order and first order differential operators given by $\mathcal{L}(t) := a^{\alpha\beta}(t) D_\alpha D_\beta$ and $\mathcal{M}^\rho(t) := b^{\alpha\rho}(t) D_\alpha$, respectively, where the summation is over $\alpha, \beta \in \{0, 1, \dots, d\}$ and where $D_\alpha = \partial/\partial x^\alpha$, for $\alpha \in \{1, \dots, d\}$, while D_0 is the identity. For each α and β we assume that $a^{\alpha\beta}(t) = a^{\alpha\beta}(t, x)$ and $b^\alpha(t) = (b^{\alpha\rho}(t, x))_{\rho=1}^{d_1}$ are real-valued $\mathcal{P} \times \mathcal{B}$ -measurable functions on $\Omega \times [0, T] \times \mathbf{R}^d$, and further that $a^{\alpha\beta}(t) = a^{\beta\alpha}(t)$ for all $t \in [0, T]$.

Equations (1.1) and (1.2) represent discrete schemes for approximating the solution to the Cauchy problem for

$$(1.3) \quad du(t, x) = (\mathcal{L}u(t, x) + f(t, x))dt + \sum_{\rho=1}^{d_1} (\mathcal{M}^\rho u(t, x) + g^\rho(t, x))dw^\rho(t)$$

for $(\omega, t, x) \in \Omega \times [0, T] \times \mathbf{R}^d$ with a given initial condition $u_0(x) = u(0, x)$. Under certain compatibility assumptions, equation (1.1) represents an implicit space-time scheme for approximating the solution to the Cauchy problem for (1.3) by replacing the differential operators with finite differences and by carrying out an implicit Euler method in time. In a similar fashion, (1.2) represents an implicit Euler method for approximating the solution to the Cauchy problem for (1.3) in time. Second order linear parabolic SPDE such as (1.3) arise in the nonlinear filtering of partially observable diffusion processes as the Zakai equation ([13, 16, 22, 1]). Since analytic solutions to (1.3) are difficult to obtain, there is a keen interest in providing accurate numerical schemes for its solution.

Our aim is to show that the strong convergence of the spatial discretization for the space-time scheme (1.1) to the solution of the Cauchy

problem for (1.3) can be accelerated to any order of accuracy with respect to the computational effort. In general, the error of finite difference approximations in the space variable for such equations is proportional to the mesh size h , for example, see [20, 21]. We show the strong convergence of the solution of the space-time scheme to the solution of the time scheme (1.2) can be accelerated to higher order accuracy by taking suitable mixtures of approximations using different mesh sizes.

This technique for obtaining higher order convergence, often referred to as *Richardson's method* after L.F. Richardson who used the idea to accelerate the convergence of finite difference schemes to deterministic partial differential equations (PDE) (see [17, 18]), falls under a broadly applicable category of extrapolation techniques, for instance see the survey articles [2, 9]. In particular, in [19, 14, 10] Richardson's method is implemented to accelerate the weak convergence of Euler approximations for stochastic differential equations. Recently, in [4] Gyöngy and Krylov considered a semi-discrete scheme for solving (1.3) which discretized via finite differences in the space variable, while allowing the scheme to vary continuously in time, and showed that the strong convergence of the spatial approximation can be accelerated by Richardson's method. The current paper extends these results to the implicit space-time scheme (1.1).

We must mention that for the present scheme one cannot also accelerate in time unless certain commutators of the differential operator \mathcal{M}^ρ in equation (1.3) vanish, see [3]. For *deterministic* PDE we plan to address the simultaneous acceleration of the convergence of approximations with respect to space and time in a future paper. Results concerning acceleration for monotone finite difference schemes for degenerate parabolic and elliptic PDE are given in [5], however our scheme is not necessarily monotone.

In the next section, we present our assumptions as well as some preliminaries. Then in Section 3 we record the main results, namely Theorems 3.1, 3.2, and 3.3, the last of which says that the convergence of the spatial approximation can be accelerated to any order of accuracy. In Section 4 we provide results which will be needed for the proofs of Theorems 3.1 and 3.2. In particular, we recall the solvability of the space-time scheme (1.1), for the convenience of the reader, and present a new contribution—an estimate for the supremum of the solution to the scheme in appropriate spaces that is independent of h , the spatial mesh size. In Section 5 we give the proof of a more general result and show that it implies Theorem 3.2 and hence Theorem 3.1.

We end with some notation that will be used throughout this work. Let $\ell^2(G_h)$ be the set of real-valued functions ϕ on G_h such that

$$|\phi|_{\ell^2(G_h)}^2 := |h|^d \sum_{x \in G_h} |\phi(x)|^2 < \infty$$

and note that this notation will also be used for functions in $\ell^2(\mathbf{R}^d)$.

For a nonnegative integer m , let $W_2^m = W_2^m(\mathbf{R}^d)$ be the usual Hilbert-Sobolev space of functions on \mathbf{R}^d with norm $\|\cdot\|_m$. We note that for $L^2 = L^2(\mathbf{R}^d) = W_2^0$ the norm will be denoted by $\|\cdot\|_0$. We use the notation $D^l\phi$ for the collection of all l th order spatial derivatives of ϕ . Let

$$\mathbf{W}_2^m(T) := L^2(\Omega \times [0, T], \mathcal{P}, W_2^m)$$

denote the space of W_2^m -valued square integrable predictable processes on $\Omega \times [0, T]$. These are the natural spaces in which to seek solutions to (1.3).

2. PRELIMINARIES AND ASSUMPTIONS

We begin by setting some assumptions on our operators and recalling well known results concerning the solvability and rates of convergence for our schemes. In particular, we will discuss an $\ell^2(G_h)$ notion of solution and an L^2 notion of solution and recall an important lemma relating these function spaces.

An L^2 -valued continuous process $u = (u(t))_{t \in [0, T]}$ is called a generalized solution to (1.3) if $u \in W_2^1$ for almost every $(\omega, t) \in \Omega \times [0, T]$,

$$\int_0^T \|u(t)\|_1^2 dt < \infty$$

almost surely, and

$$\begin{aligned} (u(t), \phi) = & \int_0^t ((a^{0\beta} - D_\alpha a^{\alpha\beta}) D_\beta u(s) + f(x), \phi) - (a^{\alpha\beta} D_\beta u, D_\alpha \phi) ds \\ & + (u_0, \phi) + \sum_{\rho=1}^{d_1} \int_0^t (\mathcal{M}^\rho u(s) + g^\rho(s), \phi) dw^\rho(s) \end{aligned}$$

holds for all $t \in [0, T]$ and $\phi \in C_0^\infty(\mathbf{R}^d)$.

Assumption 2.1. *For each $(\omega, t) \in \Omega \times [0, T]$ the functions $a^{\alpha\beta}$ are m times and the functions b^α are $m+1$ times continuously differentiable in x . Moreover there exist constants K_0, \dots, K_{m+1} such that for $l \leq m$*

$$|D^l a^{\alpha\beta}| \leq K_l$$

and for $l \leq m+1$

$$|D^l b^\alpha|_{\ell_2} \leq K_l$$

for all values of $\alpha, \beta \in \{0, \dots, d\}$ and $(\omega, t, x) \in \Omega \times [0, T] \times \mathbf{R}^d$.

Assumption 2.2. *There exists a positive constant κ such that*

$$\sum_{\alpha, \beta=1}^d (2a^{\alpha\beta} - b^{\alpha\rho} b^{\beta\rho}) z^\alpha z^\beta \geq \kappa |z|^2$$

for all $(\omega, t, x) \in \Omega \times [0, T] \times \mathbf{R}^d$, $z \in \mathbf{R}^d$, and $\rho \in \{1, \dots, d_1\}$.

Assumption 2.3. *The initial condition $u_0 \in L^2(\Omega, \mathcal{F}_0, W_2^{m+1})$, the space of W_2^{m+1} -valued square integrable \mathcal{F}_0 -measurable functions on Ω . The f and g^ρ , for $\rho \in \{1, \dots, d_1\}$, are predictable processes on $\Omega \times [0, T]$ taking values in W_2^m and W_2^{m+1} , respectively. Moreover*

$$E \int_0^T (\|f(t)\|_m^2 + \|g(t)\|_{m+1}^2) dt + E \|u_0\|_{m+1}^2 < \infty,$$

where $\|g(t)\|_l^2 := \sum_{\rho=1}^{d_1} \|g(t)^\rho\|_l^2$.

Under Assumptions 2.1, 2.2, and 2.3, the existence of a unique solution $u \in \mathbf{W}_2^{m+2}(T)$ to (1.3) is a classical result (see for example [15, 12] or Theorem 5.1 from [11]).

Remark 2.4. We note that by Sobolev's embedding of $W_2^m \subset \mathcal{C}_b$, the space of bounded continuous functions, for $m > d/2$ we can find a continuous function of x which is equal to u_0 almost everywhere for almost all $\omega \in \Omega$. Likewise, for each $(\omega, t) \in \Omega \times [0, T]$ there exists continuous functions of x which coincide with $f(t)$ and $g^\rho(t)$ for almost every $x \in \mathbf{R}^d$. Thus, if Assumption 2.3 holds with $m > d/2$ we assume that u_0 , $f(t)$, and $g^\rho(t)$ are continuous in x for all $t \in [0, T]$.

For a nonnegative integer \mathbf{m} , let $\bar{\mathbf{m}} := \mathbf{m} \vee 1$ and $\Lambda_0 := \Lambda \setminus \{0\}$. We place the following additional requirements on our space-time scheme.

Assumption 2.5. *For all $\omega \in \Omega$, for $i \in \{0, \dots, n\}$, for $\lambda, \mu \in \Lambda_0$, and for $\nu \in \Lambda$: the $\mathbf{a}^{\lambda\mu}$ are $\bar{\mathbf{m}}$ times continuously differentiable in x ; the $\mathbf{a}^{0\nu}$ and $\mathbf{a}^{\nu 0}$ are \mathbf{m} times continuously differentiable in x ; and the \mathbf{b}^ν are \mathbf{m} times continuously differentiable in x . Moreover there exist constants $A_0, \dots, A_{\bar{\mathbf{m}}}$ such that for $\lambda, \mu \in \Lambda_0$ and $j \leq \bar{\mathbf{m}}$ we have*

$$|D^j \mathbf{a}^{\lambda\mu}| \leq A_j$$

and for $\lambda \in \Lambda$ and $j \leq \mathbf{m}$ we have

$$|D^j \mathbf{a}^{\lambda 0}| \leq A_j, \quad |D^j \mathbf{a}^{0\lambda}| \leq A_j, \quad \text{and} \quad |D^j \mathbf{b}^\lambda| \leq A_j$$

for all $(\omega, x) \in \Omega \times \mathbf{R}^d$ for $i \in \{0, \dots, n\}$.

Assumption 2.6. *There exists a positive constant κ such that*

$$\sum_{\lambda, \mu \in \Lambda_0} (2\mathbf{a}^{\lambda\mu} - \mathbf{b}^{\lambda\rho} \mathbf{b}^{\mu\rho}) z_\lambda z_\mu \geq \kappa \sum_{\lambda \in \Lambda_0} z_\lambda^2$$

for all $(\omega, x) \in \Omega \times \mathbf{R}^d$, $i \in \{0, \dots, n\}$, $\rho \in \{1, \dots, d_1\}$, and numbers z_λ , $\lambda \in \Lambda_0$.

For (1.1) to be consistent with (1.3) we also require the following.

Assumption 2.7. *For $i \in \{0, \dots, n\}$*

$$\begin{aligned} \mathbf{a}_i^{00} &= a_i^{00}, \\ \sum_{\lambda \in \Lambda_0} \mathbf{a}_i^{\lambda 0} \lambda^\alpha + \sum_{\mu \in \Lambda_0} \mathbf{a}_i^{0\mu} \mu^\alpha &= a_i^{\alpha 0} + a_i^{0\alpha}, \end{aligned}$$

$$\sum_{\lambda, \mu \in \Lambda_0} \mathbf{a}_i^{\lambda\mu} \lambda^\alpha \mu^\beta = a_i^{\alpha\beta},$$

$$\mathbf{b}_i^{0\rho} = b_i^{0\rho},$$

and

$$\sum_{\lambda \in \Lambda_0} \mathbf{b}_i^{\lambda\rho} \lambda^\alpha = b_i^{\alpha\rho}$$

for all $\alpha, \beta \in \{1, \dots, d\}$ and $\rho \in \{1, \dots, d_1\}$.

Remark 2.8. If Λ_0 is a basis for \mathbf{R}^d and Assumption 2.7 holds then Assumption 2.1 implies 2.5 and 2.2 implies 2.6 with $\mathbf{m} = m$.

A solution $v^h = (v_i^h)_{i=1}^n$ to (1.1) with an $\ell^2(G_h)$ -valued \mathcal{F}_0 -measurable initial condition v_0^h is understood as a sequence of $\ell^2(G_h)$ -valued random variables satisfying (1.1) on the grid G_h . The following result is well known and we provide it for the sake of completeness.

Theorem 2.9. *Let f and g^ρ be \mathcal{F}_i -adapted $\ell^2(G_h)$ -valued processes and let v_0^h be an \mathcal{F}_0 -measurable $\ell^2(G_h)$ -valued initial condition. If Assumption 2.5 holds then (1.1) admits a unique $\ell^2(G_h)$ -valued solution for sufficiently small τ .*

Proof. By Assumption 2.5, for each $i \in \{1, \dots, n\}$, equation (1.1) is a recursion with bounded linear operators on $\ell^2(G_h)$. In particular, for each h the operator norm of τL^h is smaller than a constant less than 1 for sufficiently small τ , independently of $\omega \in \Omega$. Hence $(I - \tau L^h)$ is invertible in $\ell^2(G_h)$ for sufficiently small τ , by the invertibility of operators in a neighborhood of the (invertible) identity operator I . Therefore, for $i \in \{1, \dots, n\}$ we are guaranteed an $\ell^2(G_h)$ -valued ϕ satisfying $(I - \tau L_i^h)\phi = \psi$ for all $\psi \in \ell^2(G_h)$ and moreover this solution is easily seen to be unique. Thus we can construct a unique solution to the scheme iteratively. \square

The rate of convergence of the solution v^h of (1.1) (and v of (1.2)) to the solution u of (1.3) with initial condition u_0 is known. In [6, 7, 8], Gyöngy and Millet obtained the rate of convergence for a class of equations in the nonlinear setting of which our schemes are a special case. Namely, in the situation of Remark 2.8, if Assumptions 2.1, 2.2, and 2.3 hold with $a^{\alpha\beta}$, b^α , f , and g^ρ all Hölder continuous in time with exponent $1/2$ then

$$\begin{aligned} & E \max_{i \leq n} \sum_{|\lambda| \leq m+1} \sum_{x \in G_h} |\delta_{h,\lambda}(v_i^h(x) - u_i(x))|^2 h^d \\ & + E \tau \sum_{i=1}^n \sum_{|\lambda| \leq m+2} \sum_{x \in G_h} |\delta_{h,\lambda}(v_i^h(x) - u_i(x))|^2 h^d \leq N(h^2 + \tau) \end{aligned}$$

for sufficiently small τ , $h \in (0, 1)$, and for a constant N that is independent of h and τ . The principal interest of this paper is to investigate

higher order convergence with respect to the spatial discretization, that is, to obtain an estimate, similar to the above, with a higher power of h by applying Richardson's method.

While it is natural to seek solutions to (1.1) on the grid, carrying out our analysis on the whole space will have certain advantages when it comes to providing estimates for solutions to our schemes. Indeed, we observe that (1.1) is well defined not only on G_h but for all $x \in \mathbf{R}^d$. Therefore, we introduce an alternate notion of solution. A solution to (1.1) on $\Omega \times T_\tau \times \mathbf{R}^d$ with an L^2 -valued \mathcal{F}_0 -measurable initial condition v_0^h is a sequence $v^h = (v_i^h)_{i=1}^n$ of L^2 -valued random variables satisfying (1.1). In a similar spirit, solutions to (1.2) with the appropriate initial condition are understood as sequences of W_2^1 -valued random variables satisfying (1.2) in W_2^{-1} . The next result follows immediately from the considerations in the proof of Theorem 2.9.

Theorem 2.10. *Let f and g^ρ be \mathcal{F}_i -adapted L^2 -valued processes and let v_0^h be an \mathcal{F}_0 -measurable L^2 -valued initial condition. If Assumption 2.5 holds then (1.1) admits a unique L^2 -valued solution for sufficiently small τ .*

By Sobolev's embedding theorem, for $l > d/2$ there exists a linear operator $I : W_2^l \rightarrow C_b$ such that $\phi(x) = I\phi(x)$ for almost every $x \in \mathbf{R}^d$ and $\sup_{x \in \mathbf{R}^d} |I\phi(x)| \leq N\|\phi\|_l$ for all $\phi \in W_2^l$ where N is a constant. We recall the following useful embedding of $W_2^l \subseteq \ell^2(G_h)$ from [4].

Lemma 2.11. *Let $l > d/2$ and $|h| \in (0, 1)$. For all $\phi \in W_2^l$ the embedding*

$$(2.1) \quad \sum_{x \in G_h} |I\phi(x)|^2 |h|^d \leq N\|\phi\|_l^2$$

holds for a constant N that depends only on d and l .

Proof. For $z \in \mathbf{R}^d$ let $B_r(x) := \{x \in \mathbf{R}^d; |x - z| < r\}$. By the embedding of W_2^l into C_b , for $\phi \in C_b$ we have

$$\begin{aligned} |\phi(z)|^2 &\leq \sup_{x \in B_1(0)} \phi^2(z + hx) \\ &\leq N \sum_{|\alpha| \leq l} h^{2|\alpha|} \int_{B_1(0)} |(D^\alpha \phi)(z + hx)|^2 dx \\ &\leq N \sum_{|\alpha| \leq l} |h|^{2|\alpha| - d} \int_{B_h(z)} |(D^\alpha \phi)(x)|^2 dx \\ &\leq N|h|^{-d} \sum_{|\alpha| \leq l} \int_{B_h(z)} |(D^\alpha \phi)(x)|^2 dx \end{aligned}$$

for a constant N depending only on d and l and thus

$$|\phi|_{\ell^2(G_h)}^2 = \sum_{z \in G_h} |\phi(z)|^2 |h|^d \leq N \sum_{|\alpha| \leq l} \sum_{z \in G_h} \int_{B_h(z)} |(D^\alpha \phi)(x)|^2 dx,$$

which yields the desired embedding. \square

We will show that the restriction of a continuous modification of an L^2 -valued solution to (1.1) to the grid G_h is also a solution in the $\ell^2(G_h)$ sense. Thus we will carry out our analysis in the whole space and obtain estimates independent of h in appropriate Sobolev spaces for the L^2 -valued solutions of (1.1) and (1.2).

We provide the aforementioned Sobolev space estimates in Section 4. We then use these estimates in Section 5 to prove the main results, which are the focus of the next section.

3. MAIN RESULTS

To accelerate the convergence of the spatial approximation by Richardson's method we must have an expansion for the solution v^h to (1.1) with initial data $v_0^h = u_0$ in powers of the mesh size h . This relies on the possibility of proving the existence of sequences of random fields $v^{(0)}(x), v^{(1)}(x), \dots, v^{(k)}(x)$, for $x \in \mathbf{R}^d$ and integer $k \geq 0$, satisfying certain properties. Namely, $v^{(0)}, \dots, v^{(k)}$ are independent of h ; $v^{(0)}$ is the solution of (1.2) with initial value u_0 ; and an expansion

$$(3.1) \quad v_i^h(x) = \sum_{j=0}^k \frac{h^j}{j!} v_i^{(j)}(x) + R_i^{\tau,h}(x)$$

holds almost surely for $i \in \{1, \dots, n\}$ and $x \in G_h$, where $R^{\tau,h}$ is an $\ell_2(G_h)$ -valued adapted process such that

$$(3.2) \quad E \max_{i \leq n} \sup_{x \in G_h} |R_i^{\tau,h}(x)|^2 \leq N h^{2(k+1)} \mathcal{K}_m$$

for

$$\mathcal{K}_m := E \|u_0\|_{m+1}^2 + E \tau \sum_{i=0}^n (\|f_i\|_m^2 + \|g_i\|_{m+1}^2) < \infty$$

and a constant N independent of τ and h .

Our first result concerns the existence of such an expansion.

Theorem 3.1. *If Assumptions 2.1, 2.2, 2.3, 2.5, 2.6, and 2.7 hold with*

$$\mathbf{m} = m > k + 1 + \frac{d}{2}$$

for an integer $k \geq 0$ then expansion (3.1) and estimate (3.2) hold for a constant N depending only on $d, d_1, \Lambda, m, K_0, \dots, K_{m+1}, A_0, \dots, A_m, \kappa$, and T .

In the proof of Theorem 3.1, as v^h is defined not only on G_h but for all $x \in \mathbf{R}^d$, we will see that one can replace G_h in (3.2) with \mathbf{R}^d . We also note that in the situation of Remark 2.8, if Assumptions 2.1 and 2.2 hold with $m > k + 1 + d/2$ then the conditions of Theorem 3.1 are satisfied.

Taking differences of expansion (3.1) clearly yields

$$\delta_{h,\lambda} v_i^h(x) = \sum_{j=0}^k \frac{h^j}{j!} \delta_{h,\lambda} v_i^{(j)}(x) + \delta_{h,\lambda} R_i^{\tau,h}(x)$$

for any $\lambda = (\lambda_1, \dots, \lambda_p) \in \Lambda^p$, for integer $p \geq 0$, where $\Lambda^0 := \{0\}$ and $\delta_{h,\lambda} := \delta_{h,\lambda_1} \times \dots \times \delta_{h,\lambda_p}$. The bound on $\delta_{h,\lambda} R_i^{\tau,h}$ is not obvious, nevertheless we have the following generalization of the above theorem.

Theorem 3.2. *If the assumptions of Theorem 3.1 hold with*

$$\mathbf{m} = m > k + p + 1 + \frac{d}{2}$$

for a nonnegative integer p then for $\lambda \in \Lambda^p$ expansion (3.1) and

$$E \max_{i \leq n} \sup_{x \in G_h} |\delta_{h,\lambda} R_i^{\tau,h}(x)|^2 + E \max_{i \leq n} |h|^d \sum_{x \in G_h} |\delta_{h,\lambda} R_i^{\tau,h}(x)|^2 \leq N h^{2(k+1)} \mathcal{K}_m$$

hold for a constant N depending only on $d, d_1, \Lambda, m, K_0, \dots, K_{m+1}, A_0, \dots, A_m, \kappa$, and T .

The proof of Theorems 3.1 and 3.2 appear in Section 5 following the considerations in the next section. Currently we shall discuss how to implement Richardson's method to obtain higher order convergence in the spatial approximation, extending the result from [4] to the space-time scheme.

Fix an integer $k \geq 0$ and let

$$(3.3) \quad \bar{v}^h := \sum_{j=0}^k \beta_j v^{2^{-j}h}$$

where $v^{2^{-j}h}$ solves, with $2^{-j}h$ in place of h , the space-time scheme (1.1) with initial condition u_0 . Here β is given by $(\beta_0, \beta_1, \dots, \beta_k) := (1, 0, \dots, 0)V^{-1}$ where V^{-1} is the inverse of the Vandermonde matrix with entries $V^{ij} := 2^{-(i-1)(j-1)}$ for $i, j \in \{1, \dots, k+1\}$. Recall that $v^{(0)}$ is the solution to (1.2) with initial condition u_0 .

Theorem 3.3. *Under the assumptions of Theorem 3.1,*

$$(3.4) \quad E \max_{i \leq n} \sup_{x \in G_h} |\bar{v}_i^h(x) - v_i^{(0)}(x)|^2 \leq N |h|^{2(k+1)} \mathcal{K}_m$$

for a constant N depending only on $d, d_1, \Lambda, m, K_0, \dots, K_{m+1}, A_0, \dots, A_m, \kappa$, and T .

Proof. By Theorem 3.1 we have the expansion

$$v^{2^{-j}h} = v^{(0)} + \sum_{i=1}^k \frac{h^i}{i! 2^{ij}} v^{(i)} + \hat{r}^{\tau, 2^{-j}h} h^{k+1}$$

for each $j \in \{0, 1, \dots, k\}$ where $\hat{r}^{\tau, 2^{-j}h} := h^{-(k+1)} R^{\tau, 2^{-j}h}$. Then

$$\begin{aligned} \bar{v}^h &= \left(\sum_{j=0}^k \beta_j \right) v^{(0)} + \sum_{j=0}^k \sum_{i=1}^k \beta_j \frac{h^i}{i! 2^{ij}} v^{(i)} + \sum_{j=0}^k \beta_j \hat{r}^{\tau, 2^{-j}h} h^{k+1} \\ &= v^{(0)} + \sum_{i=1}^k \frac{h^i}{i!} v^{(i)} \sum_{j=0}^k \frac{\beta_j}{2^{ij}} + \sum_{j=0}^k \beta_j \hat{r}^{\tau, 2^{-j}h} h^{k+1} \\ &= v^{(0)} + \sum_{j=0}^k \beta_j \hat{r}^{\tau, 2^{-j}h} h^{k+1} \end{aligned}$$

since $\sum_{j=0}^k \beta_j = 1$ and $\sum_{j=0}^k \beta_j 2^{-ij} = 0$ for each $i \in \{1, 2, \dots, k\}$ by the definition of $(\beta_0, \dots, \beta_k)$. Now using the bound on $R^{\tau, h}$ from Theorem 3.1 together with this last calculation yields the desired result. \square

One can also construct rapidly converging approximations of derivatives of $v^{(0)}$. That is, if the conditions of Theorem 3.1 hold instead with

$$\mathbf{m} = m > k + p + 1 + \frac{d}{2}$$

for nonnegative integers k and p then Theorem 3.3 holds with $\delta_{h, \lambda} \bar{v}^h$ and $\delta_{h, \lambda} v^{(0)}$ in place of \bar{v}^h and $v^{(0)}$, respectively, for $\lambda \in \Lambda^p$. Therefore, using suitable linear combinations of finite differences of \bar{v}^h one can construct rapidly converging approximations for the derivatives of $v^{(0)}$.

In the next section, we present material that will be used to prove the main results in this section. In particular, we provide estimates for the L^2 -valued solutions of (1.1) and (1.2) in appropriate Sobolev spaces.

4. AUXILIARY RESULTS

We include the following bound, which is given for the continuous time case in [4], for the convenience of the reader.

Lemma 4.1. *If Assumptions 2.5 and 2.6 hold then for all $\phi \in L^2$*

$$\begin{aligned} Q_i(\phi) &:= \int_{\mathbf{R}^d} 2\phi(x) L_i^h \phi(x) + \sum_{\rho=1}^{d_1} |M_i^{h, \rho} \phi(x)|^2 dx \\ &\leq N \|\phi\|_0^2 - \frac{\kappa}{2} \sum_{\lambda \in \Lambda_0} \|\delta_{h, \lambda} \phi\|_0^2 \end{aligned}$$

for all $i \in \{1, \dots, n\}$ and for a constant N depending only on κ , A_0 , A_1 , and the cardinality of Λ .

Proof. First observe that for $\mu \in \Lambda_0$ the conjugate operator in L_2 to $\delta_{-h, \mu}$ is $-\delta_{h, \mu}$. Notice also that

$$\delta_{h, \mu}(\phi\psi) = \phi\delta_{h, \mu}\psi + (T_{h, \mu}\psi)\delta_{h, \mu}\phi$$

where $T_{h,\mu}\psi(x) = \psi(x + h\mu)$. Thus by simple calculations $Q = Q^{(1)} + Q^{(2)} + Q^{(3)} + Q^{(4)}$ where

$$Q_i^{(1)}(\phi) := - \int_{\mathbf{R}^d} \sum_{\lambda, \mu \in \Lambda_0} ((2\mathbf{a}_i^{\lambda\mu} - \mathbf{b}_i^{\lambda\rho} \mathbf{b}_i^{\mu\rho})(\delta_{h,\lambda}\phi)\delta_{h,\mu}\phi)(x) dx,$$

$$Q_i^{(2)}(\phi) := -2 \int_{\mathbf{R}^d} \sum_{\lambda, \mu \in \Lambda_0} ((T_{h,\mu}\phi)(\delta_{h,\lambda}\phi)\delta_{h,\mu}\mathbf{a}_i^{\lambda\mu})(x) dx,$$

$$Q_i^{(3)}(\phi) := 2 \int_{\mathbf{R}^d} (\mathbf{a}_i^{00}\phi^2(x) + \phi(x) \sum_{\lambda \in \Lambda_0} (\mathbf{a}_i^{\lambda 0}\delta_{h,\lambda}\phi + \mathbf{a}_i^{0\lambda}\delta_{-h,\lambda}\phi)(x)) dx,$$

and

$$Q_i^{(4)}(\phi) := \int_{\mathbf{R}^d} (\mathbf{b}_i^{00}\phi^2(x) + 2 \sum_{\lambda \in \Lambda_0} \mathbf{b}_i^{\lambda\rho} \mathbf{b}_i^{0\rho} \phi \delta_{h,\lambda}\phi(x)) dx.$$

By Assumption 2.6,

$$Q_i^{(1)}(\phi) \leq -\kappa \sum_{\lambda \in \Lambda_0} \|\delta_{h,\lambda}\phi\|_0^2$$

and by Assumption 2.5, Young's inequality, and the shift invariance of Lebesgue measure,

$$Q_i^{(j)}(\phi) \leq \frac{\kappa}{6} \sum_{\lambda \in \Lambda_0} \|\delta_{h,\lambda}\phi\|_0^2 + N\|\phi\|_0^2$$

for each $j \in \{2, 3, 4\}$ with a constant N depending only on the cardinality of Λ , κ , A_0 and, for $j = 2$, also on A_1 . \square

We also recall the following discrete Gronwall lemma. Note that we use the convention that summation over an empty set is zero.

Lemma 4.2. *For constants $K \in (0, 1)$ and C , if $(a_i)_{i=0}^n$ is a nonnegative sequence such that $a_j \leq C + K \sum_{i=1}^j a_i$ holds for each $j \in \{0, \dots, n\}$ then $a_j \leq C(1 - K)^{-j}$ for $j \in \{0, \dots, n\}$.*

Proof. Let $b_j = C + K \sum_{i=1}^j b_i$ and note that $(1 - K)b_j = b_{j-1}$. Then $a_j \leq b_j$ for $j \leq n$ by induction, since $a_0 \leq C = b_0$ and

$$a_j(1 - K) \leq C + \sum_{i=1}^{j-1} a_i \leq C + \sum_{i=1}^{j-1} b_i = b_j(1 - K),$$

assuming $a_{j-1} \leq b_{j-1}$. Therefore $a_j \leq b_j = C(1 - K)^{-j}$ for each $j \leq n$ and for $K \in (0, 1)$. \square

The following provides a Sobolev space estimate for solutions to the space-time scheme that is independent of h . For an integer $m \geq 0$, denote by $\mathbf{W}_2^m(\tau)$ the space of W_2^m -valued predictable processes ϕ on $\Omega \times T_\tau$ such that

$$\llbracket \phi \rrbracket_m := E\tau \sum_{i=1}^n \|\phi_i\|_m^2 < \infty$$

and note that we write

$$\llbracket g \rrbracket_m = E\tau \sum_{i=1}^n \sum_{\rho=1}^{d_1} \|g_i^\rho\|_m^2$$

for functions $g = (g^\rho)_{\rho=1}^{d_1}$.

Theorem 4.3. *For $\mu \in \Lambda$ and $\rho \in \{1, \dots, d_1\}$, let $f^\mu, g^\rho \in \mathbf{W}_2^{\mathbf{m}}(\tau)$. If Assumption 2.5 holds then for each nonzero h there exists a unique solution $\nu \in \mathbf{W}_2^{\mathbf{m}}(\tau)$ of*

$$(4.1) \quad \nu_i = \nu_{i-1} + \sum_{\mu \in \Lambda} (L_i^h \nu_i + f_i^\mu) \tau + \sum_{\rho=1}^{d_1} (M_{i-1}^{h,\rho} \nu_{i-1} + g_{i-1}^\rho) \xi_i^\rho$$

for any $W_2^{\mathbf{m}+1}$ -valued \mathcal{F}_0 -measurable initial condition ν_0 . Further, if Assumption 2.6 is also satisfied then

$$(4.2) \quad E \max_{i \leq n} \|\nu_i\|_{\mathbf{m}}^2 + E\tau \sum_{i=1}^n \sum_{\lambda \in \Lambda} \|\delta_{h,\lambda} \nu_i\|_{\mathbf{m}}^2 \leq NE\tau \|\nu_0\|_{\mathbf{m}+1}^2 + NE\tau \sum_{i=0}^n (\|f_i\|_{\mathbf{m}}^2 + \|g_i\|_{\mathbf{m}}^2)$$

holds for a constant N that depends only on $d, d_1, \mathbf{m}, \Lambda, A_0, \dots, A_{\mathbf{m}}, \kappa$, and T .

Proof. By Theorem 2.10, the existence of a unique sequence of L^2 -valued random variables solving (1.1) is known. For $f^\mu, g^\rho \in \mathbf{W}_2^{\mathbf{m}}(\tau)$ and an $W_2^{\mathbf{m}+1}$ -valued initial condition ν_0 , there exists a unique sequence of $W_2^{\mathbf{m}}$ -valued random variables satisfying (4.1). The estimate (4.2) can be achieved easily with a constant N depending on h , so in particular the solution is in $\mathbf{W}_2^{\mathbf{m}}(\tau)$.

Next we prove the estimate (4.2) for a constant independent of h . For convenience we denote $\mathfrak{K}_{\mathbf{m}}^n := \tau \sum_{i=0}^n (\|f_i\|_{\mathbf{m}}^2 + \|g_i\|_{\mathbf{m}}^2)$. Considering equalities of the form $a^2 + b^2 = 2a(a-b) - |a-b|^2$ we note that (4.1) implies

$$\begin{aligned} \|\nu_i\|_0^2 - \|\nu_{i-1}\|_0^2 &= 2(\nu_i, \nu_i - \nu_{i-1}) - \|\nu_i - \nu_{i-1}\|_0^2 \\ &= 2(\nu_i, L_i^h \nu_i + f_i^\mu) \tau + 2(\nu_{i-1}, M_{i-1}^{h,\rho} \nu_{i-1} + g_{i-1}^\rho) \xi_i^\rho \\ &\quad + 2(\nu_i - \nu_{i-1}, M_{i-1}^{h,\rho} \nu_{i-1} + g_{i-1}^\rho) \xi_i^\rho - \|\nu_i - \nu_{i-1}\|_0^2 \\ &= 2(\nu_i, L_i^h \nu_i + f_i^\mu) \tau + 2(\nu_{i-1}, M_{i-1}^{h,\rho} \nu_{i-1} + g_{i-1}^\rho) \xi_i^\rho \\ &\quad + \|(M_{i-1}^{h,\rho} \nu_{i-1} + g_{i-1}^\rho) \xi_i^\rho\|_0^2 - \|L_i^h \nu_i + f_i^\mu\|_0^2 \tau^2 \end{aligned}$$

where here and in what follows we suppress the sums over $\mu \in \Lambda_0$ and $\rho \in \{1, \dots, d_1\}$. Summing up over i , we have

$$(4.3) \quad \|\nu_j\|_0^2 \leq \|\nu_0\|_0^2 + \mathcal{H}_j + \mathcal{I}_j + \mathcal{J}_j$$

where

$$\mathcal{H}_j := \sum_{i=1}^j 2(\nu_i, L_i^h \nu_i + f_i^\mu) \tau,$$

$$\mathcal{I}_j := \sum_{i=1}^j 2(\nu_{i-1}, M_{i-1}^{h,\rho} \nu_{i-1} + g_{i-1}^\rho) \xi_i^\rho,$$

and

$$\mathcal{J}_j := \sum_{i=1}^j \|(M_{i-1}^{h,\rho} \nu_{i-1} + g_{i-1}^\rho) \xi_i^\rho\|_0^2.$$

By an application of Itô's formula, it is easy to see that for $\pi, \rho \in \{1, \dots, d_1\}$

$$\xi_{i+1}^\pi \xi_{i+1}^\rho = (\Delta w^\pi(t_i))(\Delta w^\rho(t_i)) = Y_{i+1}^{\pi\rho} - Y_i^{\pi\rho} + \tau \delta_{\pi\rho}$$

for all $i \in \{1, \dots, n\}$ where

$$Y^{\pi\rho}(t) := \int_0^t (w^\pi(s) - w_{\gamma(s)}^\pi) dw^\rho(s) + \int_0^t (w^\rho(s) - w_{\gamma(s)}^\rho) dw^\pi(s),$$

$\gamma(s)$ is the piecewise defined function taking value $\gamma(s) = i$ for $s \in [i\tau, (i+1)\tau)$, and $\delta_{\pi\rho} = 1$ when $\pi = \rho$ and 0 otherwise. Thus can write $\mathcal{J}_j = \mathcal{J}_j^{(1)} + \mathcal{J}_j^{(2)}$ where

$$\mathcal{J}_j^{(1)} := \sum_{i=1}^j \sum_{\rho=1}^{d_1} \|M_{i-1}^{h,\rho} \nu_{i-1} + g_{i-1}^\rho\|_0^2 \tau$$

and

$$\mathcal{J}_j^{(2)} := \int_0^{t_j} \sum_{\pi, \rho=1}^{d_1} (M_{\gamma(s)}^{h,\pi} \nu_{\gamma(s)} + g_{\gamma(s)}^\pi, M_{\gamma(s)}^{h,\rho} \nu_{\gamma(s)} + g_{\gamma(s)}^\rho) dY^{\pi\rho}(s).$$

Then we note that since Lemma 4.1 holds for all $t \in [0, T]$, in particular

$$\begin{aligned} \mathcal{H}_j + \mathcal{J}_j^{(1)} &\leq \tau \sum_{\lambda \in \Lambda} \|\delta_{h,\lambda} \nu_0\|_0^2 + \tau \sum_{i=1}^j \left(Q_i(\nu_i) + (\nu_i, f_i^\mu) + \frac{\tau}{\varepsilon} \|g_{i-1}\|_0^2 \right) \\ &\leq C\tau \|\nu_0\|_1^2 + N\tau \sum_{i=1}^j \|\nu_i\|_0^2 - \frac{\kappa}{2} \tau \sum_{i=1}^j \sum_{\lambda \in \Lambda} \|\delta_{h,\lambda} \nu_i\|_0^2 + N\mathfrak{K}_0^j \end{aligned}$$

where here $\|\nu_0\|_0^2 \leq C\|\nu_0\|_1^2$ and N is a positive constant depending only on κ, A_0, A_1 , and the cardinality of Λ . Thus we can replace equation (4.3) by

$$(4.4) \quad \begin{aligned} \|\nu_j\|_0^2 + \tau \sum_{i=1}^j \sum_{\lambda \in \Lambda} \|\delta_{h,\lambda} \nu_i\|_0^2 &\leq N\tau \|\nu_0\|_1^2 + N\tau \sum_{i=1}^j \|\nu_i\|_0^2 \\ &\quad + N\mathfrak{K}_0^j + \mathcal{I}_j + \mathcal{J}_j^{(2)} \end{aligned}$$

for a constant N that depends only on κ , A_0 , A_1 , C , and the cardinality of Λ .

Next we observe that

$$E\mathcal{I}_j = \sum_{i=1}^j 2 \int_{\mathbf{R}^d} E \left(E \left(\nu_{i-1} (M_{i-1}^{h,\rho} \nu_{i-1} + g_{i-1}^\rho) \xi_i^\rho \mid \mathcal{F}_{i-1} \right) \right) dx = 0$$

since ξ_{i+1}^ρ is independent of \mathcal{F}_i and ν_i , $M_i^{h,\rho} \nu_i$, and g_i^ρ are all \mathcal{F}_i -measurable for $i \in \{0, \dots, n\}$. Similarly, we see that $E\mathcal{J}_j^{(2)} = 0$ since the expectation of the stochastic integral is zero. Therefore, taking the expectation of (4.4) we have that

$$(4.5) \quad \begin{aligned} E\|\nu_j\|_0^2 + E\tau \sum_{i=1}^n \sum_{\lambda \in \Lambda} \|\delta_{h,\lambda} \nu_i\|_0^2 &\leq NE\tau \|\nu_0\|_1^2 + NE\mathfrak{K}_0^n \\ &\quad + NE\tau \sum_{i=1}^j \|\nu_i\|_0^2 \end{aligned}$$

for each $j \in \{1, \dots, n\}$. Excluding for the time being the difference term on the left hand side of (4.5) and applying Lemma 4.2 we obtain

$$E\|\nu_j\|_0^2 \leq N(E\tau \|\nu_0\|_1^2 + E\mathfrak{K}_0^n)(1 - N\tau)^{-j}$$

and, since $(1 - N\tau)^{-j} = (1 - N\frac{T}{n})^{-j} \leq (1 - N\frac{T}{n})^{-n} \leq C'e^{NT}$, we have

$$(4.6) \quad \max_{i \leq n} E\|\nu_i\|_0^2 \leq NE\tau \|\nu_0\|_1^2 + NE\mathfrak{K}_0^n$$

for a constant N here that depends only on the parameters κ , A_0 , A_1 , T , and the cardinality of Λ . Using equation (4.6) we can eliminate the last term on the right hand side of (4.5). In particular, we have

$$E\tau \sum_{i=1}^n \sum_{\lambda \in \Lambda} \|\delta_{h,\lambda} \nu_i\|_0^2 \leq NE\tau \|\nu_0\|_1^2 + NE\mathfrak{K}_0^n.$$

Using the Davis inequality we can bound $\max |\mathcal{J}^{(2)}|$ and $\max |\mathcal{I}|$. Namely,

$$\begin{aligned}
& E \max_{i \leq n} |\mathcal{J}_j^{(2)}| \\
& \leq 3 \sum_{\pi, \rho=1}^{d_1} E \left\{ \int_0^T \|M_{\gamma(s)}^{h, \rho} \nu_{\gamma(s)} + g_{\gamma(s)}^\rho\|_0^2 \|M_{\gamma(s)}^{h, \pi} \nu_{\gamma(s)} + g_{\gamma(s)}^\pi\|_0^2 d\langle Y^{\pi\rho} \rangle(s) \right\}^{1/2} \\
& \leq C \sum_{\pi, \rho=1}^{d_1} E \left\{ \int_0^T \|M_{\gamma(s)}^{h, \rho} \nu_{\gamma(s)} + g_{\gamma(s)}^\rho\|_0^4 |w^\pi(s) - w_{\gamma(s)}^\pi|^2 ds \right\}^{1/2} \\
& \leq C \sum_{\pi, \rho=1}^{d_1} E \left(\max_{i \leq n} \sqrt{\tau} \|M_i^{h, \rho} \nu_i + g_i^\rho\|_0 \right. \\
& \quad \left. \times \left\{ \frac{1}{\tau} \int_0^T \|M_{\gamma(s)}^{h, \rho} \nu_{\gamma(s)} + g_{\gamma(s)}^\rho\|_0^2 |w^\pi(s) - w_{\gamma(s)}^\pi|^2 ds \right\}^{1/2} \right)
\end{aligned}$$

where C is a constant independent of τ and h that is allowed to change from one instance to the next. Therefore,

$$\begin{aligned}
(4.7) \quad & E \max_{i \leq n} |\mathcal{J}_i^{(2)}| \leq d_1 C \sum_{\rho=1}^{d_1} \tau E \max_{i \leq n} \|M_i^{h, \rho} \nu_i + g_i^\rho\|_0^2 \\
& + \frac{C}{\tau} \sum_{\pi, \rho=1}^{d_1} E \int_0^T \|M_{\gamma(s)}^{h, \rho} \nu_{\gamma(s)} + g_{\gamma(s)}^\rho\|_0^2 |w^\pi(s) - w_{\gamma(s)}^\pi|^2 ds
\end{aligned}$$

by Young's inequality. The first term on the right hand side of (4.7) is bounded from above by the sum over all $i \in \{1, \dots, n\}$, hence

$$\begin{aligned}
& \sum_{\rho=1}^{d_1} \tau E \max_{i \leq n} \|M_i^{h, \rho} \nu_i + g_i^\rho\|_0^2 \leq E \tau \sum_{i=0}^n \|M_i^{h, \rho} \nu_i + g_i^\rho\|_0^2 \\
& \leq C E \tau \sum_{i=0}^n \left(\sum_{\lambda \in \Lambda} \|\delta_{h, \lambda} \nu_i\|_0^2 + \|g_i\|_0^2 \right),
\end{aligned}$$

and the second term on the right hand side of (4.7) yields

$$\begin{aligned}
& \frac{1}{\tau} \sum_{\pi, \rho=1}^{d_1} E \int_0^T \|M_{\gamma(s)}^{h, \rho} \nu_{\gamma(s)} + g_{\gamma(s)}^\rho\|_0^2 |w^\pi(s) - w_{\gamma(s)}^\pi|^2 ds \\
& \leq \frac{1}{\tau} \sum_{\pi, \rho=1}^{d_1} E \left(E \left(\int_0^T \|M_{\gamma(s)}^{h, \rho} \nu_{\gamma(s)} + g_{\gamma(s)}^\rho\|_0^2 |w^\pi(s) - w_{\gamma(s)}^\pi|^2 ds \mid \mathcal{F}_{\gamma(s)} \right) \right) \\
& \leq E \tau \sum_{i=0}^n \left(\sum_{\lambda \in \Lambda} \|\delta_{h, \lambda} \nu_i\|_0^2 + \|g_i\|_0^2 \right)
\end{aligned}$$

by the tower property for conditional expectations. Combining these estimates we see that $E \max |\mathcal{J}^{(2)}|$ is estimated by terms already appearing on the right hand side of (4.4) for a constant N that depends also on d_1 .

Similarly, we note that

$$\begin{aligned} \mathcal{I}_j &= \sum_{i=1}^j 2(\nu_{i-1}, M_{i-1}^{h,\rho} \nu_{i-1} + g_{i-1}^\rho) \Delta w_{i-1}^\rho \\ &= 2 \int_0^{t_j} (\nu_{\gamma(s)}, M_{\gamma(s)}^{h,\rho} \nu_{\gamma(s)} + g_{\gamma(s)}^\rho) dw^\rho(s). \end{aligned}$$

Applying the Davis inequality

$$\begin{aligned} E \max_{i \leq n} |\mathcal{I}_i| &\leq 6 \sum_{\rho=1}^{d_1} E \left\{ \int_0^T \|\nu_{\gamma(s)}\|_0^2 \|M_{\gamma(s)}^{h,\rho} \nu_{\gamma(s)} + g_{\gamma(s)}^\rho\|_0^2 ds \right\}^{1/2} \\ &\leq 6 \sum_{\rho=1}^{d_1} E \left(\max_{i \leq n} \|\nu_i\|_0 \left\{ \int_0^T \|M_{\gamma(s)}^{h,\rho} \nu_{\gamma(s)} + g_{\gamma(s)}^\rho\|_0^2 ds \right\}^{1/2} \right) \end{aligned}$$

and then Young's inequality

$$(4.8) \quad E \max_{i \leq n} |\mathcal{I}_i| \leq \frac{1}{2} E \max_{i \leq n} \|\nu_i\|_0^2 + CE\tau \sum_{i=0}^n \left(\sum_{\lambda \in \Lambda} \|\delta_{h,\lambda} \nu_i\|_0^2 + \|g_i\|_0^2 \right)$$

we see that $E \max |\mathcal{I}|$ is also estimated by terms already appearing on the right and side of (4.4).

Returning to (4.4) and taking the maximum followed by the expectation we have

$$E \max_{i \leq n} \|\nu_i\|_0^2 + E\tau \sum_{i=1}^n \sum_{\lambda \in \Lambda} \|\delta_{h,\lambda} \nu_i\|_0^2 \leq NE\tau \|\nu_0\|_1^2 + NE\mathfrak{K}_0^n,$$

using (4.6) and the estimates on $E \max |\mathcal{J}^{(2)}|$ and $E \max |\mathcal{I}|$. Thus (4.2) holds when $\mathbf{m} = 0$.

If $\mathbf{m} \geq 1$ we differentiate (4.1) with respect to x^l and introduce the notation $\tilde{\phi}$ for the derivative of a function ϕ in the direction x^l for $l \in \{1, \dots, d\}$. Then (4.1) becomes

$$\begin{aligned} (4.9) \quad \tilde{\nu}_i &= \tilde{\nu}_{i-1} + \sum_{\lambda, \mu \in \Lambda} (\mathfrak{a}_i^{\lambda\mu} \delta_{h,\lambda} \delta_{-h,\mu} \tilde{\nu}_i + \hat{f}_i^\mu) \tau \\ &\quad + \sum_{\rho=1}^{d_1} \sum_{\lambda \in \Lambda} (\mathfrak{b}_{i-1}^{\lambda\rho} \delta_{h,\lambda} \tilde{\nu}_{i-1} + \hat{g}_{i-1}^\rho) \xi_i^\rho \end{aligned}$$

where $\hat{f}^\mu := \tilde{f}^\mu$ for nonzero μ , $\hat{f}^0 := \tilde{f}^0 + \tilde{\mathfrak{a}}^{\lambda\mu} \delta_{h,\lambda} \delta_{-h,\mu} \nu$ and $\hat{g}^\rho := \tilde{g}^\rho + \tilde{\mathfrak{b}}^{\lambda\rho} \delta_{h,\lambda} \nu$. Recalling that $\partial_\mu = \mu^l D_l$ for $\mu \in \Lambda_0$, we proceed as

before but now using the inequality

$$\begin{aligned} E\tau \sum_{i=1}^n (\tilde{\nu}_i, \delta_{h,\lambda} \delta_{-h,\mu} \nu_i) &\leq \tau \sum_{i=1}^n E \|\tilde{\nu}_i\|_0 \|\delta_{h,\lambda} \partial_\mu \nu_i\|_0 \\ &\leq \varepsilon E\tau \sum_{i=1}^n \|D\delta_{h,\lambda} \nu_i\|_0^2 + \frac{N}{\varepsilon} E\tau \sum_{i=1}^n \|\tilde{\nu}_i\|_0^2 \end{aligned}$$

which holds for arbitrary $\varepsilon > 0$ and N depending only on $|\mu|$. This leads to the following

$$\begin{aligned} (4.10) \quad E\|\tilde{\nu}_j\|_0^2 + E\tau \sum_{i=1}^j \sum_{\lambda \in \Lambda} \|\delta_{h,\lambda} \nu_i\|_0^2 &\leq NE\tau \|\nu_0\|_1^2 + NE\mathfrak{K}_1^n \\ &\quad + \frac{1}{2d} E\tau \sum_{i=1}^j \sum_{\lambda \in \Lambda} \|D\delta_{h,\lambda} \nu_i\|_0^2 + NE\tau \sum_{i=1}^j \|\tilde{\nu}_i\|_0^2 \end{aligned}$$

for each x^l , $l \in \{1, \dots, d\}$. Summing up over each direction x^l , the term with factor $1/2d$ can be seen to be estimated by other terms already appearing on the right hand side of (4.10). Then by the same procedure as before we obtain

$$(4.11) \quad E \max_{i \leq n} \|D\nu_i\|_0^2 + E\tau \sum_{i=1}^n \sum_{\lambda \in \Lambda} \|D\delta_{h,\lambda} \nu_i\|_0^2 \leq NE\tau \|\nu_0\|_1^2 + NE\mathfrak{K}_1^n$$

which proves the theorem when $\mathbf{m} = 1$.

Assuming that $\mathbf{m} \geq 2$ and that (4.2) holds for each integer $p < \mathbf{m}$ in place of \mathbf{m} , then we can differentiate (4.1) $(p+1)$ times and, repurposing the notation $\tilde{\phi}$ for the $(p+1)$ th order derivatives of ϕ with respect to x , we obtain (4.9) with different \hat{f}^0 and \hat{g}^ρ . Namely, the \hat{f}^0 will be the sum of \tilde{f}^0 and linear combinations of certain i th order derivatives of $\mathbf{a}^{\lambda\mu}$ together with certain $(p+1-i)$ th order derivatives of $\delta_{h,\lambda} \delta_{-h,\mu} \nu$, for integer $i \leq (p+1)$. As before, the L^2 -norms of the $(p+1-i)$ th derivatives of $\delta_{h,\lambda} \delta_{-h,\mu} \nu$ are dominated by the L^2 -norms of the $(p+2-i)$ th derivatives of $\delta_{h,\lambda} \nu$ which are in turn less than the W_2^{p+1} -norm of $\delta_{h,\lambda} \nu$. After similar changes are made in \hat{g}^ρ we obtain the counterpart of (4.11) which then yields (4.2) with $(p+1)$ in place of \mathbf{m} . \square

We also have the following Sobolev space estimate for solutions to the implicit time scheme. Let m be a nonnegative integer.

Theorem 4.4. *Let $f \in \mathbf{W}_2^m(\tau)$ and $g^\rho \in \mathbf{W}_2^{m+1}(\tau)$. If Assumptions 2.1 and 2.2 hold then (1.2) has a unique solution $v \in \mathbf{W}_2^{m+2}(\tau)$ for a*

given W_2^{m+1} -valued \mathcal{F}_0 -measurable initial condition v_0 . Moreover

$$\begin{aligned} E \max_{i \leq n} \|v_i\|_{m+1}^2 + E \tau \sum_{i=1}^n \|v_i\|_{m+2}^2 &\leq NE \|v_0\|_{m+1}^2 \\ &+ NE \tau \sum_{i=0}^n (\|f_i\|_m^2 + \|g_i\|_{m+1}^2) \end{aligned}$$

holds for a constant N depending only on $d, d_1, m, K_0, \dots, K_{m+1}, \kappa$, and T .

Proof. Proving the solvability of (1.2) reduces to solving the elliptic problem

$$(I - \tau \mathcal{L}_i)v_i = v_{i-1} + \tau f_i + \sum_{\rho=1}^{d_1} \xi_i^\rho (\mathcal{M}_{i-1}^\rho v_{i-1} + g_{i-1}^\rho)$$

for each $i \in \{1, \dots, n\}$ where I is the identity. That is, we claim that $\mathcal{A} := (I - \tau \mathcal{L})$ is a W_2^m -valued operator on W_2^{m+2} such that \mathcal{A}_i is

- (i) *bounded*, i.e. $\|\mathcal{A}_i \phi\|_m^2 \leq K \|\phi\|_{m+2}^2$ for a constant K ,
- (ii) and *coercive*, i.e. $\langle \mathcal{A}_i \phi, \phi \rangle \geq \lambda \|\phi\|_{m+2}^2$ for a constant $\lambda > 0$,

for every $i \in \{1, \dots, n\}$ and for all $\phi \in W_2^{m+2}$, where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between W_2^{m+2} and W_2^m based on the inner product in W_2^{m+1} . Then by the separability of W_2^{m+2} , there exists a countable dense subset $\{e_j\}_{j=1}^\infty$ such that for fixed $p \geq 1$, $\phi_p = \sum_{j=1}^p c_j e_j$ for constants c_j where $\phi_p \in W_2^{m+2}$ is not identically zero. We fix i and for every $\psi \in W_2^m$ consider \mathcal{A} acting on ϕ_p , that is $\langle \mathcal{A} \phi_p, e_j \rangle = \langle \psi, e_j \rangle$ for all $j \in \{1, \dots, p\}$. Taking linear combinations we obtain $\langle \mathcal{A} \phi_p, \phi_p \rangle = \langle \psi, \phi_p \rangle$ from which we derive

$$\lambda \|\phi_p\|_{m+2}^2 \leq \langle \mathcal{A} \phi_p, \phi_p \rangle = \langle \psi, \phi_p \rangle \leq \|\psi\|_m \|\phi_p\|_{m+2}$$

by the coercivity of \mathcal{A} and an application of the Cauchy-Schwarz inequality. Thus $\|\phi_p\|_{m+2} \leq \frac{1}{\lambda} \|\psi\|_m$ and hence (by the reflexivity of W_2^{m+2}), there exists a subsequence p_k such that ϕ_{p_k} converges weakly to ϕ and in particular $\langle \mathcal{A}_i \phi_{p_k}, e_j \rangle \rightarrow \langle \mathcal{A} \phi, e_j \rangle$ for every j . Therefore for every $\psi \in W_2^m$ there exists a $\phi \in W_2^{m+2}$ satisfying $\mathcal{A}_i \phi = \psi$ for every $i \in \{1, \dots, n\}$. Moreover, this solution is easily seen to be unique.

Using the existence and uniqueness to the elliptic problem in each interval, we note that

$$v_0 + \tau f_1 + \sum_{\rho=1}^{d_1} (\mathcal{M}_0^\rho v_0 + g_0^\rho) \xi_1^\rho \in W_2^m$$

by Assumption 2.3 and therefore there exists a $v_1 \in W_2^{m+2}$ satisfying

$$(I - \tau \mathcal{L}_1)v_1 = v_0 + \tau f_1 + \sum_{\rho=0}^{d_1} (\mathcal{M}_0^\rho v_0 + g_0^\rho) \xi_1^\rho.$$

Further, assuming that there exists a $v_i \in W_2^{m+2}$ satisfying (1.2) we have that

$$v_i + \tau f_{i+1} + \sum_{\rho=0}^{d_1} (\mathcal{M}_i^\rho v_i + g_i^\rho) \xi_{i+1}^\rho \in W_2^m$$

by the induction hypothesis and Assumption 2.3, and therefore there exists a $v_{i+1} \in W_2^{m+2}$ satisfying (1.2). Hence we obtain $v = (v_i)_{i=1}^n$ such that each $v_i \in W_2^{m+2}$ satisfies (1.2).

It only remains to prove the claim concerning ellipticity of \mathcal{A} . By Assumption 2.1, clearly \mathcal{A}_i is a bounded linear operator for each i . We see that

$$\begin{aligned} \langle \mathcal{A}\phi, \phi \rangle &= \langle I\phi, \phi \rangle - \tau \langle \mathcal{L}\phi, \phi \rangle \\ &= \|\phi\|_{m+1}^2 - \tau \langle \mathcal{L}\phi, \phi \rangle \end{aligned}$$

where, by Assumptions 2.1 and 2.2,

$$\begin{aligned} \langle \mathcal{L}\phi, \phi \rangle &= ((a^{0\beta} - D_\alpha a^{\alpha\beta}) D_\beta \phi, \phi) - (a^{\alpha\beta} D_\beta \phi, D_\alpha \phi) \\ &\leq C \|\phi\|_{m+1}^2 - \frac{\kappa}{2} \|\phi\|_{m+2}^2 \end{aligned}$$

in the W_2^{m+1} inner product for $\alpha, \beta \in \{1, \dots, d\}$ and for a constant C depending on K_0 and K_1 . Therefore

$$\langle \mathcal{A}\phi, \phi \rangle \geq \frac{\kappa}{2} \tau \|\phi\|_{m+2}^2 + (1 - \tau K) \|\phi\|_{m+1}^2 \geq \frac{\kappa}{2} \tau \|\phi\|_{m+2}^2$$

for sufficiently small τ and hence (ii) is satisfied.

To prove the estimate, we use a method similar to that in the proof of Theorem 4.3 to arrive at (4.3) with v in place of ν , \mathcal{L} in place of L^h , \mathcal{M}^ρ in place of $M^{h,\rho}$, all in the W_2^{m+1} -norm instead of the L^2 -norm. Again, we decompose \mathcal{J} into $\mathcal{J}^{(1)}$ and $\mathcal{J}^{(2)}$ using the processes $Y^{\pi\rho}(t)$ that arise by applying the Itô formula to the product of increments of the independent Wiener processes. However this time, instead of using Lemma 4.1, we observe that

$$\mathcal{H}_j + \mathcal{J}_j^{(1)} \leq N\tau \sum_{i=1}^j \|v_i\|_{m+1}^2 - \frac{\kappa}{2} \tau \sum_{i=1}^j \|v_i\|_{m+2}^2 + N\tau \sum_{i=0}^j (\|f_i\|_m^2 + \|g_i\|_{m+1}^2)$$

since

$$\int_{\mathbf{R}^d} 2v(x) \mathcal{L}v(x) + \sum_{\rho=1}^{d_1} |\mathcal{M}^\rho v(x)|^2 dx \leq (\varepsilon - \kappa C) \|v\|_{m+2}^2 + C \|v\|_{m+1}^2$$

for $\varepsilon > 0$ by the considerations above. Therefore we have that

$$\begin{aligned} \|v_j\|_{m+1}^2 + \tau \sum_{i=1}^j \|v_i\|_{m+2}^2 &\leq N \|v_0\|_{m+1}^2 + N \mathcal{I}_j + N \mathcal{J}_j^{(2)} \\ &\quad + N\tau \sum_{i=0}^j (\|f_i\|_m^2 + \|g_i\|_{m+1}^2) \end{aligned}$$

and the estimate follows by considering the maximum and then taking the expectation. Moreover, with the estimate, it is clear that the solution $v \in \mathbf{W}_2^{m+2}(\tau)$. \square

We will use the theorem above to obtain estimates in appropriate Sobolev spaces for a system of time discretized equations. For $i \in \{0, \dots, n\}$ and an integer $p \geq 1$, let

$$\mathcal{L}_i^{(0)} := \sum_{\lambda, \mu \in \Lambda} \mathfrak{a}_i^{\lambda\mu} \partial_\lambda \partial_\mu,$$

$$\mathcal{M}_i^{(0)\rho} := \sum_{\lambda \in \Lambda} \mathfrak{b}_i^{\lambda\rho} \partial_\lambda,$$

and let

$$\begin{aligned} \mathcal{L}_i^{(p)} := p! \sum_{\lambda, \mu \in \Lambda_0} \mathfrak{a}_i^{\lambda\mu} \sum_{j=0}^p A_{p,j} \partial_\lambda^{j+1} \partial_\mu^{p-j+1} + (p+1)^{-1} \sum_{\lambda \in \Lambda_0} \mathfrak{a}_i^{\lambda 0} \partial_\lambda^{p+1} \\ + (p+1)^{-1} \sum_{\mu \in \Lambda_0} \mathfrak{a}_i^{0\mu} \partial_\mu^{p+1}, \end{aligned}$$

$$\mathcal{M}_i^{(p)\rho} := (p+1)^{-1} \sum_{\lambda \in \Lambda_0} \mathfrak{b}_i^{\lambda\rho} \partial_\lambda^{p+1},$$

$$\mathcal{O}_i^{h(p)} := L_i^h - \sum_{j=0}^p \frac{h^j}{j!} \mathcal{L}_i^{(j)},$$

and

$$\mathcal{R}_i^{h(p)\rho} := M_i^{h,\rho} - \sum_{j=0}^p \frac{h^j}{j!} \mathcal{M}_i^{(j)\rho}$$

where $A_{p,j}$ is defined by

$$(4.12) \quad A_{p,j} = \frac{(-1)^{p-j}}{(j+1)!(p-j+1)!}.$$

For $p \geq 1$, the values of $\mathcal{L}^{(p)}\phi$ and $\mathcal{M}^{(p)\rho}\phi$ are obtained by formally taking the p th derivatives in h of $L^h\phi$ and $M^{h,\rho}\phi$ at $h = 0$.

For a positive integer $k \leq m$, the sequences of random fields $v^{(1)}, \dots, v^{(k)}$ needed in (3.1) will be the embeddings of random variables taking values in certain Sobolev spaces obtained as solutions to a system of time discretized SPDE. Namely, as the solutions to

$$\begin{aligned} (4.13) \quad \nu_i^{(p)} = \nu_{i-1}^{(p)} + (\mathcal{L}_i \nu_i^{(p)} + \sum_{l=1}^p C_p^l \mathcal{L}_i^{(l)} \nu_i^{(p-l)})\tau \\ + (\mathcal{M}_{i-1}^\rho \nu_{i-1}^{(p)} + \sum_{l=1}^p C_p^l \mathcal{M}_{i-1}^{(l)\rho} \nu_{i-1}^{(p-l)})\xi_i^\rho, \end{aligned}$$

for $p \in \{1, \dots, k\}$ where $C_p^l = p(p-1) \cdots (p-l+1)/l!$ is the binomial coefficient and $\nu^{(0)}$ is the solution to (1.2) from Theorem 4.4.

Theorem 4.5. *Let Assumptions 2.1, 2.2, 2.3, and 2.5 hold with $\mathbf{m} = m \geq k \geq 1$ and let $\nu^{(0)} \in \mathbf{W}_2^{m+2}(\tau)$ be the solution to (1.2) with initial condition u_0 from Theorem 4.4. Then the system (4.13) with initial condition*

$$\nu_0^{(1)} = \nu_0^{(2)} = \dots = \nu_0^{(k)} = 0$$

has a unique set of solutions $(\nu^{(p)})_{p=1}^k$ such that each $\nu^{(p)} \in \mathbf{W}_2^{m+2-p}(\tau)$. Moreover for each $p \in \{1, \dots, k\}$,

$$(4.14) \quad \begin{aligned} & E \max_{i \leq n} \|\nu_i^{(p)}\|_{m+1-p}^2 + E\tau \sum_{i=1}^n \|\nu_i^{(p)}\|_{m+2-p}^2 \\ & \leq NE\tau \sum_{i=0}^n (\|f_i\|_m^2 + \|g_i\|_{m+1}^2) \end{aligned}$$

holds for a constant N depending only on $d, d_1, \Lambda, m, K_0, \dots, K_{m+1}, A_0, \dots, A_m, \kappa$, and T .

Proof. For convenience let

$$F_i^{(p)} := \sum_{j=1}^p C_j^p \mathcal{L}_i^{(j)} \nu_i^{(p-j)}$$

and

$$G_i^{(p)\rho} := \sum_{j=1}^p C_j^p \mathcal{M}_i^{(j)\rho} \nu_i^{(p-j)}$$

where we write $G^{(p)} = \sum_{\rho=1}^{d_1} G^{(p)\rho}$.

Observe that for each $p \in \{1, \dots, k\}$ the equation for $\nu^{(p)}$ in (4.13) depends only on $\nu^{(l)}$ for $l \leq p$ and does not involve any of the unknown processes $\nu^{(l)}$ with indices $l > p$. Therefore we shall prove the solvability of the system and the desired properties on $\nu^{(p)}$ recursively using Theorem 4.4.

For $p = 1$, we have

$$(4.15) \quad \nu_i^{(1)} = \nu_{i-1}^{(1)} + (\mathcal{L}_i \nu_i^{(1)} + F_i^{(1)})\tau + \sum_{\rho=1}^{d_1} (\mathcal{M}_{i-1}^\rho \nu_{i-1}^{(1)} + G_{i-1}^{(1)\rho}) \xi_i^\rho.$$

Since $\nu^{(0)} \in \mathbf{W}_2^{m+2}(\tau)$, for $\mathbf{m} = m$ we have that $F^{(1)} \in \mathbf{W}_2^{m-1}(\tau)$ and $G^{(1)} \in \mathbf{W}_2^m(\tau)$ by Assumption 2.5. Hence by Theorem 4.4, there exists a unique $\nu^{(1)} \in \mathbf{W}_2^{m+1}(\tau)$ satisfying (4.15) with initial condition $\nu_0^{(1)} = 0$. Further, $\nu^{(1)}$ is estimated by (4.14) and thus Theorem 4.5 holds with $p = 1$.

Now we assume that for $m \geq k \geq 2$ and $p \in \{2, \dots, k\}$ we have unique $\nu^{(1)}, \dots, \nu^{(p-1)}$ solving (4.13) for $\nu^{(1)} = \dots = \nu^{(p-1)} = 0$ with the desired properties. In particular, observe that for $j \in \{1, \dots, p\}$

$$(4.16) \quad \llbracket \mathcal{L}^{(j)} \nu^{(p-j)} \rrbracket_{m-p} \leq N \llbracket \nu^{(p-j)} \rrbracket_{m+2-(p-j)}$$

and for each $\rho \in \{1, \dots, d_1\}$

$$(4.17) \quad \llbracket \mathcal{M}^{(i)\rho} \nu^{(p-j)} \rrbracket_{m-p+1} \leq N \llbracket \nu^{(p-j)} \rrbracket_{m+1-(p-j)}$$

for a constant N . Therefore it follows that $F^{(p)} \in \mathbf{W}_2^{m-p}(\tau)$ and $G^{(p)} \in \mathbf{W}_2^{m-p+1}(\tau)$. Applying Theorem 4.4 yields the existence of a unique solution $\nu^{(p)} \in \mathbf{W}_2^{m-p+2}(\tau)$ that satisfies (4.13) with initial condition $\nu_0^{(p)} = 0$. Together with (4.16) and (4.17) the estimate from Theorem 4.4 implies that (4.14) holds. Further, the uniqueness of each $\nu^{(p)}$ follows from Theorem 4.4. \square

For the convenience of the reader we record the following lemma and two remarks from [4] that will be used in proving the error estimates.

Lemma 4.6. *Let $\phi \in W_2^{p+1}$ and $\psi \in W_2^{p+2}$ for a nonnegative integer p and let $\lambda, \mu \in \Lambda_0$. Set*

$$\partial_\lambda \phi = \lambda^j D_j \phi \quad \text{and} \quad \partial_{\lambda\mu} = \partial_\lambda \partial_\mu.$$

Then we have

$$(4.18) \quad \frac{\partial^p}{(\partial h)^p} \delta_{h,\lambda} \phi(x) = \int_0^1 \theta^p \partial_\lambda^{p+1} \phi(x + h\theta\lambda) d\theta$$

and

$$(4.19) \quad \begin{aligned} & \frac{\partial^p}{(\partial h)^p} \delta_{h,\lambda} \delta_{-h,\mu} \psi(x) \\ &= \int_0^1 \int_0^1 (\theta_1 \partial_\lambda - \theta_2 \partial_\mu)^p \partial_{\lambda\mu} \psi(x + h(\theta_1 \lambda - \theta_2 \mu)) d\theta_1 d\theta_2 \end{aligned}$$

for almost all $x \in \mathbf{R}^d$ for each $h \in \mathbf{R}$. Furthermore, for integer $l \geq 0$ if $\phi \in W_2^{p+2+l}$ and $\psi \in W_2^{p+3+l}$ then

$$(4.20) \quad \left\| \delta_{h,\lambda} \phi - \sum_{j=0}^p \frac{h^j}{(j+1)!} \partial_\lambda^{j+1} \phi \right\|_l \leq \frac{|h|^{p+1}}{(p+2)!} \|\partial_\lambda^{p+2} \phi\|_l$$

and

$$(4.21) \quad \left\| \delta_{h,\lambda} \delta_{-h,\mu} \psi - \sum_{i=0}^p h^i \sum_{j=0}^i A_{i,j} \partial_\lambda^{j+1} \partial_\mu^{i-j+1} \psi \right\|_l \leq N |h|^{p+1} \|\psi\|_{l+p+3},$$

where $A_{i,j}$ is defined by (4.12) and N depends on λ, μ, d , and p .

Proof. It suffices to prove the lemma for $\phi, \psi \in C_0^\infty(\mathbf{R}^d)$. For $p = 0$, formula (4.18) is obtained by applying the Newton-Leibniz formula to $\phi(x + \theta h\lambda)$ as a function of $\theta \in [0, 1]$. Namely,

$$\phi(x + h\lambda) - \phi(x) = \int_{u=x}^{u=x+h\lambda} D_j \phi(u) du = h \int_{\theta=0}^{\theta=1} \lambda^j D_j \phi(x + \theta h\lambda) d\theta$$

and therefore $\delta_{h,\lambda} \phi(x) = \int_0^1 \partial_\lambda \phi(x + \theta h\lambda) d\theta$. Applying the Newton-Leibniz formula again yields (4.19) with $p = 0$. After that, for $p \geq 1$

one obtains (4.18) and (4.19) by differentiating both parts of these equations written with $p = 1$.

Next by Taylor's formula for smooth $f(h)$ we have

$$f(h) = \sum_{j=0}^p \frac{h^j}{j!} \frac{d^j}{(dh)^j} f(0) + \frac{1}{p!} \int_0^h (h - \theta)^p \frac{d^{p+1}}{(dh)^{p+1}} f(\theta) d\theta.$$

Applying this to

$$\delta_{h,\lambda} \phi(x) = \int_0^1 \partial_\lambda \phi(x + \theta h \lambda) d\theta$$

as a function of h we see that

$$\begin{aligned} \delta_{h,\lambda} \phi(x) &= \sum_{j=0}^p \frac{h^j}{(j+1)!} \partial_\lambda^{j+1} \phi(x) \\ &\quad + \frac{h^{p+1}}{p!} \int_0^1 \int_0^1 (1 - \theta_2)^p \theta_1^{p+1} \partial_\lambda^{p+2} \phi(x + h \theta_1 \theta_2 \lambda) d\theta_1 d\theta_2. \end{aligned}$$

Now to prove (4.20), it remains only to use that by Minkowski's integral inequality the W_2^l -norm of the last term is less than the W_2^l -norm of $\partial_\lambda^{p+2} \phi$ times

$$\frac{|h|^{p+1}}{p!} \int_0^1 \int_0^1 (1 - \theta_2)^p \theta_1^{p+1} d\theta_1 d\theta_2 = \frac{|h|^{p+1}}{(p+2)!}.$$

Similarly, by observing that the value at $h = 0$ of the right hand side of (4.19) is

$$p! \sum_{j=0}^p A_{p,j} \partial_\lambda^{j+1} \partial_\mu^{p-j+1} \psi(x),$$

we see that the left hand side of (4.21) is the W_2^l -norm of

$$\frac{h^{p+1}}{p!} \int_0^1 \int_0^1 \int_0^1 (1 - \theta_3)^p (\theta_1 \partial_\lambda - \theta_2 \partial_\mu)^{p+1} \partial_{\lambda\mu} \psi(x + h \theta_3 (\theta_1 \lambda - \theta_2 \mu)) d\theta_1 d\theta_2 d\theta_3,$$

which yields (4.21). \square

For integers $l \geq 0$ and $r \geq 1$, denote by $W_{h,2}^{l,r}$ the Hilbert space of functions ϕ on \mathbf{R}^d such that

$$(4.22) \quad \|\phi\|_{l,r,h}^2 := \sum_{\lambda_1, \dots, \lambda_r \in \Lambda} \|\delta_{h,\lambda_1} \times \dots \times \delta_{h,\lambda_r} \phi\|_l^2 < \infty$$

and set $W_{h,2}^{l,0} = W_2^l$. Then for any $\phi \in W_2^{l+r}$ we have

$$\|\phi\|_{l,r,h} \leq N \|\phi\|_{l+r},$$

where N depends only on $|\Lambda_0|^2 := \sum_{\lambda \in \Lambda_0} |\lambda|^2$ and r .

Remark 4.7. Formula (4.18) with $p = 0$ and Minkowski's integral inequality imply that

$$\|\delta_{h,\lambda}\phi\|_0 \leq \|\partial_\lambda\phi\|_0.$$

By applying this inequality to finite differences of ϕ and using induction we can conclude that $W_2^{l+r} \subset W_{h,2}^{l,r}$.

Remark 4.8. Owing to Assumption 2.7, for $i \in \{0, \dots, n\}$ we have that $\mathcal{L}_i^{(0)} = \mathcal{L}_i$ and $\mathcal{M}_i^{(0)\rho} = \mathcal{M}_i^\rho$. Also by Lemma 4.6 and Assumptions 2.5 and 2.6, for $\phi \in W_2^{p+2+l}$ and $\psi \in W_2^{p+3+l}$ we have

$$\|\mathcal{O}^{h(p)}\psi\|_l \leq N|h|^{p+1}\|\psi\|_{l+p+3}$$

and

$$\|\mathcal{R}^{h(p)\rho}\phi\|_l \leq N|h|^{p+1}\|\phi\|_{l+p+2}$$

for a constant N depending only on p, d, l, A_0, \dots, A_l , and Λ .

For integers $k, l \geq 0$, let $\nu^{(0)}, \nu^{(1)}, \dots, \nu^{(k)}$ be the functions from Theorem 4.5. We define

$$(4.23) \quad r_i^{\tau,h} := \nu_i^h - \nu_i^{(0)} - \sum_{j=1}^k \frac{h^j}{j!} \nu_i^{(j)}$$

for $i \in \{1, \dots, n\}$ where ν^h is the unique L^2 -valued solution to (1.1) that exists by Theorem 4.3 with initial condition u_0 , data $f^0 = f$ and $f^\mu = 0$, $\mu \in \Lambda_0$.

Lemma 4.9. *Let Assumptions 2.1, 2.2, 2.3, and 2.5 hold with $\mathfrak{m} = m = l + k + 1$ for integers $k, l \geq 0$ and let $r^{\tau,h}$ be defined as in equation (4.23). Then $r_0^{\tau,h} = 0$, $r^{\tau,h} \in \mathbf{W}_2^{m-k}(\tau)$ and*

$$r_i^{\tau,h} = r_{i-1}^{\tau,h} + (L_i^h r_i^{\tau,h} + F_i^{\tau,h})\tau + \sum_{\rho=1}^{d_1} (M_{i-1}^{h,\rho} r_{i-1}^{\tau,h} + G_{i-1}^{\tau,h,\rho}) \xi_i^\rho$$

for $i \in \{1, \dots, n\}$ where

$$F_i^{\tau,h} := \sum_{j=0}^k \frac{h^j}{j!} \mathcal{O}_i^{h(k-j)} \nu_i^{(j)}$$

and

$$G_{i-1}^{\tau,h,\rho} := \sum_{j=0}^k \frac{h^j}{j!} \mathcal{R}_{i-1}^{h(k-j)\rho} \nu_{i-1}^{(j)}$$

and, moreover, $F^{\tau,h} \in \mathbf{W}_2^l(\tau)$ and $G^{\tau,h,\rho} \in \mathbf{W}_2^{l+1}(\tau)$.

Proof. By Theorem 4.3 the solution to the space-time scheme $\nu^h \in \mathbf{W}_2^m(\tau)$ and by Theorem 4.4 the solution to the time scheme $\nu^{(0)} \in \mathbf{W}_2^{m+2}(\tau)$. Therefore $r^{\tau,h} \in \mathbf{W}_2^m(\tau)$ when $k = 0$ and, by Theorem 4.5, $r^{\tau,h} \in \mathbf{W}_2^{m-k}(\tau)$ when $k \geq 1$.

Observe that

$$\begin{aligned}
\sum_{i=0}^k \frac{h^i}{i!} \sum_{j=0}^{k-i} \frac{h^j}{j!} \mathcal{L}^{(j)} \nu^{(i)} &= \sum_{i=0}^{k-1} \frac{h^i}{i!} \sum_{j=1}^{k-i} \frac{h^j}{j!} \mathcal{L}^{(j)} \nu^{(i)} \\
&= \sum_{i=1}^k \sum_{j=0}^{k-i} \frac{h^{i+j}}{i!j!} \mathcal{L}^{(i)} \nu^{(j)} \\
&= \sum_{i=1}^k \sum_{j=i}^k \frac{h^j}{i!(j-i)!} \mathcal{L}^{(i)} \nu^{(j-i)} \\
&= \sum_{i=1}^k \sum_{j=1}^i \frac{h^i}{j!(i-j)!} \mathcal{L}^{(j)} \nu^{(i-j)} =: I^{\tau,h}
\end{aligned}$$

where summations over empty sets are zero. Therefore, we can rewrite $F^{\tau,h}$ as

$$F^{\tau,h} = L^h \nu^{(0)} - \mathcal{L} \nu^{(0)} + \sum_{j=1}^k \frac{h^j}{j!} L^h \nu^{(j)} - \sum_{j=1}^k \frac{h^j}{j!} \mathcal{L} \nu^{(j)} - I^{\tau,h}.$$

Similarly, observe that

$$\sum_{i=0}^k \frac{h^i}{i!} \sum_{j=0}^{k-i} \frac{h^j}{j!} \mathcal{M}^{(j)\rho} \nu^{(i)} = \sum_{i=1}^k \sum_{j=1}^i \frac{h^i}{j!(i-j)!} \mathcal{M}^{(j)\rho} \nu^{(i-j)} =: J^{\tau,h,\rho}$$

and therefore

$$G^{\tau,h,\rho} = M^{h,\rho} \nu^{(0)} - \mathcal{M} \rho \nu^{(0)} + \sum_{j=1}^k \frac{h^j}{j!} M^{h,\rho} \nu^{(j)} - \sum_{j=1}^k \frac{h^j}{j!} \mathcal{M} \rho \nu^{(j)} - J^{\tau,h,\rho}.$$

Thus, following from Remark 4.8 and Theorem 4.5, $F^{\tau,h} \in \mathbf{W}_2^l(\tau)$ and $G^{\tau,h,\rho} \in \mathbf{W}_2^{l+1}(\tau)$. \square

With the previous considerations, we are now prepared to prove the main results.

5. PROOF OF MAIN RESULTS

We prove a slightly more general result which implies Theorem 3.2. Here we suppose that $\mathbf{m} = m$.

Theorem 5.1. *Let Assumptions 2.1, 2.2, 2.3, 2.5, 2.6, and 2.7 hold with $m = l + k + 1$ for integers $l, k \geq 0$. Then for $r^{\tau,h}$ as defined in (4.23) we have*

$$(5.1) \quad E \max_{i \leq n} \|r_i^{\tau,h}\|_l^2 + E \tau \sum_{i=1}^n \sum_{\lambda \in \Lambda} \|\delta_{h,\lambda} r_i^{\tau,h}\|_l^2 \leq N |h|^{2(k+1)} \mathcal{K}_m,$$

where N depends only on $d, d_1, \Lambda, m, K_0, \dots, K_{m+1}, A_0, \dots, A_m, \kappa$, and T .

Proof. Recall that by Lemma 4.9 we have that $F^{\tau,h} \in \mathbf{W}_2^l(\tau)$ and $G^{\tau,h,\rho} \in \mathbf{W}_2^{l+1}(\tau)$. Then the left-hand-side of (5.1) is dominated by

$$(5.2) \quad NE\tau \sum_{i=1}^n (\|F_i^{\tau,h}\|_l^2 + \|G_i^{\tau,h,\rho}\|_l^2)$$

due to Lemma 4.9 and Theorem 4.3. To estimate (5.2) we observe that for $j \leq k$, by Remark 4.8 we have that

$$\|\mathcal{O}_i^{h(k-j)} \nu_i^{(j)}\|_l \leq N|h|^{k-j+1} \|\nu_i^{(j)}\|_{l+k-j+3} = N|h|^{k-j+1} \|\nu_i^{(j)}\|_{m+2-j},$$

and combining this result with Theorem 4.5 yields

$$E\tau \sum_{i=1}^n \|F_i^{\tau,h}\|_l^2 \leq N|h|^{2(k+1)} \mathcal{K}_m.$$

The bound on $G^{\tau,h,\rho}$ can be obtained in a similar fashion, yielding the desired result. \square

Now set $R^{\tau,h} := Ir^{\tau,h}$ where I is the embedding operator from Lemma 2.11. We have the following corollary to Theorem 5.1 which implies Theorem 3.2.

Corollary 5.2. *If the assumptions of Theorem 5.1 hold with $l > p+d/2$ for a nonnegative integer p then for $\lambda \in \Lambda^p$*

$$E \max_{i \leq n} \sup_{x \in \mathbf{R}^d} |\delta_{h,\lambda} R_i^{\tau,h}(x)|^2 \leq Nh^{2(k+1)} \mathcal{K}_m$$

and

$$E \max_{i \leq n} \sum_{x \in G_h} |\delta_{h,\lambda} R_i^{\tau,h}(x)|^2 |h|^d \leq Nh^{2(k+1)} \mathcal{K}_m$$

hold for a constant N depending only on $d, d_1, \Lambda, m, K_0, \dots, K_{m+1}, A_0, \dots, A_m, \kappa$, and T .

Proof. Using Sobolev's embedding of W_2^{l-p} into C_b and Remark 4.7, Theorem 5.1 implies

$$\begin{aligned} E \max_{i \leq n} \sup_{x \in \mathbf{R}^d} |\delta_{h,\lambda} R_i^{\tau,h}(x)|^2 &\leq CE \max_{i \leq n} \|r_i^{\tau,h}\|_{l-p,p,h}^2 \\ &\leq C'E \max_{i \leq n} \|r_i^{\tau,h}\|_l^2 \\ &\leq Nh^{2(k+1)} \mathcal{K}_m \end{aligned}$$

where C and C' are constants depending only on m and d , and N is a constant depending only on $m, d, d_1, \kappa, \Lambda, K_0, \dots, K_{m+1}$, and T . Similarly, by Lemma 2.11 above and Remark 4.7,

$$\begin{aligned} E \max_{i \leq n} \sum_{x \in G_h} |\delta_{h,\lambda} R_i^{\tau,h}(x)|^2 |h|^d &\leq CE \max_{i \leq n} \|\delta_{h,\lambda} R_i^{\tau,h}\|_{l-p}^2 \\ &\leq C'E \max_{i \leq n} \|r_i^{\tau,h}\|_l^2 \\ &\leq Nh^{2(k+1)} \mathcal{K}_m. \end{aligned}$$

□

For the $I : W_2^l \rightarrow C_b$ from Lemma 2.11, Theorem 3.2 follows by considering the embeddings $\hat{v}^h := I\nu^h$, where ν^h is the unique L^2 -valued solution to (1.1) with initial condition u_0 , and $v^{(j)} := I\nu^{(j)}$ for $j \in \{0, \dots, k\}$, where $\nu^{(0)}$ is the unique L^2 -valued solution to (1.2) with initial condition u_0 and the processes $\nu^{(1)}, \dots, \nu^{(k)}$ are the solutions to the system of time discretized SPDE (4.13) as given in Theorem 4.5. By Theorem 4.3, ν^h is \mathcal{F}_i -adapted and W_2^l -valued for all $i \in \{1, \dots, n\}$. For each $j \in \{1, \dots, k\}$ the $\nu^{(j)}$ are W^{p+1+k} -valued processes by Theorem 4.5. Since $l > d/2$ and $p+1-k > d/2$ the processes \hat{v}^h and $v^{(j)}$ are well defined and clearly (4.23) implies (3.1) with \hat{v}^h in place of v^h . That is, we have the expansion for a continuous version of the L^2 -valued solution.

To see that Theorem 3.2 indeed follows from Corollary 5.2 we must show that the restriction of the L^2 -valued solution to the grid G_h , a set of Lebesgue measure zero, is indeed equal almost surely to the unique $\ell^2(G_h)$ -valued solution that one would naturally obtain from (1.1). That is, we must show that

$$(5.3) \quad \hat{v}_i^h(x) = v_i^h(x)$$

almost surely for all $i \in \{1, \dots, n\}$ and for each $x \in G_h$ where v^h is the unique \mathcal{F}_i -adapted $\ell^2(G_h)$ -valued solution of (1.1) from Theorem 2.9. Therefore, for a compactly supported nonnegative smooth function ϕ on \mathbf{R}^d with unit integral and for a fixed $x \in G_h$ we define

$$\phi_\varepsilon(y) := \phi\left(\frac{y-x}{\varepsilon}\right)$$

for $y \in \mathbf{R}^d$ and $\varepsilon > 0$. Recall, by Remark 2.4, that we can obtain versions of u_0 , f , and g^ρ that are continuous in x . Since \hat{v}^h is a L^2 -valued solution of (1.1) for each ε , almost surely

$$\begin{aligned} \int_{\mathbf{R}^d} \hat{v}_i^h(y) \phi_\varepsilon(y) dy &= \int_{\mathbf{R}^d} \hat{v}_{i-1}(y) \phi_\varepsilon(y) dy + \tau \int_{\mathbf{R}^d} (L_i^h \hat{v}_i^h + f_i)(y) \phi_\varepsilon(y) dy \\ &\quad + \sum_{\rho=1}^{d_1} \xi_i^\rho \int_{\mathbf{R}^d} (M_{i-1}^{h,\rho} \hat{v}_{i-1}^h + g_{i-1}^\rho)(y) \phi_\varepsilon(y) dy \end{aligned}$$

for each $i \in \{1, \dots, n\}$. Letting $\varepsilon \rightarrow 0$, we see that both sides converge for all $i \in \{1, \dots, n\}$ and $\omega \in \Omega$. Therefore almost surely

$$\hat{v}_i^h(x) = \hat{v}_{i-1}^h(x) + (L_i^h \hat{v}_i^h(x) + f_i(x))\tau + \sum_{\rho=1}^{d_1} (M_{i-1}^{h,\rho} \hat{v}_{i-1}^h(x) + g_{i-1}^\rho(x))\xi_i^\rho$$

for all $i \in \{1, \dots, n\}$. Moreover by Lemma 2.11, the restriction of \hat{v}^h , the continuous version of ν^h , onto G_h is an $\ell^2(G_h)$ -valued process. Hence (5.3) holds, due to the uniqueness of the $\ell^2(G_h)$ -valued

\mathcal{F}_i -adapted solution of (1.1) for any $\ell^2(G_h)$ -valued \mathcal{F}_0 -measurable initial data. This finishes the proof of Theorem 3.2.

We end with the following generalization of Theorem 3.3.

Theorem 5.3. *If the assumptions of Theorem 3.2 hold with $p = 0$ and \bar{v}^h as defined in (3.3) then*

$$\begin{aligned} & E \max_{i \leq n} \sup_{x \in G_h} |\bar{v}_i^h(x) - v_i^{(0)}(x)|^2 \\ & + E \max_{i \leq n} \sum_{x \in G_h} |\bar{v}_i^h(x) - v_i^{(0)}(x)|^2 |h|^d \leq N |h|^{2(k+1)} \mathcal{K}_m \end{aligned}$$

for a constant N depending only on $d, d_1, \Lambda, m, K_0, \dots, K_{m+1}, A_1, \dots, A_m, \kappa$, and T .

This follows from Theorem 5.1 and the definition of \bar{v}^h .

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REFERENCES

1. Alan Bain and Dan Crisan, *Fundamentals of stochastic filtering*, Stochastic Modelling and Applied Probability, vol. 60, Springer, New York, 2009.
2. C. Brezinski, *Convergence acceleration during the 20th century*, J. Comput. Appl. Math. **122** (2000), no. 1-2, 1–21, Numerical analysis 2000, Vol. II: Interpolation and extrapolation.
3. A. M. Davie and J. G. Gaines, *Convergence of numerical schemes for the solution of parabolic stochastic partial differential equations*, Math. Comp. **70** (2000), no. 233, 121–134 (electronic).
4. István Gyöngy and Nicolai Krylov, *Accelerated finite difference schemes for linear stochastic partial differential equations in the whole space*, SIAM J. Math. Anal. **42** (2010), no. 5, 2275–2296.
5. ———, *Accelerated finite difference schemes for second order degenerate elliptic and parabolic problems in the whole space*, Math. Comp. **80** (2011), no. 275, 1431–1458.
6. István Gyöngy and Annie Millet, *On discretization schemes for stochastic evolution equations*, Potential Anal. **23** (2005), no. 2, 99–134.
7. ———, *Rate of convergence of implicit approximations for stochastic evolution equations*, Stochastic differential equations: theory and applications, Interdiscip. Math. Sci., vol. 2, World Sci. Publ., Hackensack, NJ, 2007, pp. 281–310.
8. ———, *Rate of convergence of space time approximations for stochastic evolution equations*, Potential Anal. **30** (2009), no. 1, 29–64.
9. D. C. Joyce, *Survey of extrapolation processes in numerical analysis*, SIAM Rev. **13** (1971), 435–490.
10. P. E. Kloeden, E. Platen, and N. Hofmann, *Extrapolation methods for the weak approximation of Itô diffusions*, SIAM J. Numer. Anal. **32** (1995), no. 5, 1519–1534.

11. N. V. Krylov, *An analytic approach to SPDEs*, Stochastic partial differential equations: six perspectives, Math. Surveys Monogr., vol. 64, Amer. Math. Soc., Providence, RI, 1999, pp. 185–242.
12. N. V. Krylov and B. L. Rozovskii, *The Cauchy problem for linear stochastic partial differential equations*, Math. USSR, Izv. **11** (1977), no. 6, 1267–1284.
13. Hiroshi Kunita, *Cauchy problem for stochastic partial differential equations arising in nonlinear filtering theory*, Systems Control Lett. **1** (1981/82), no. 1, 37–41.
14. Paul Malliavin and Anton Thalmaier, *Numerical error for SDE: asymptotic expansion and hyperdistributions*, C. R. Math. Acad. Sci. Paris **336** (2003), no. 10, 851–856.
15. E. Pardoux, *Équations aux dérivées partielles stochastiques de type monotone*, Séminaire sur les Équations aux Dérivées Partielles (1974–1975), III, Exp. No. 2, Collège de France, Paris, 1975, p. 10.
16. ———, *Stochastic partial differential equations and filtering of diffusion processes*, Stochastics **3** (1979), no. 2, 127–167.
17. L.F. Richardson, *The approximate arithmetical solution by finite differences of physical problems involving differential equations, with an application to the stresses in a masonry dam*, Philosophical Transactions of the Royal Society of London. Series A, Containing Papers of a Mathematical or Physical Character **210** (1911), 307–357.
18. L.F. Richardson and J.A. Gaunt, *The deferred approach to the limit. part i. single lattice. part ii. interpenetrating lattices*, Philosophical Transactions of the Royal Society of London. Series A, containing papers of a mathematical or physical character **226** (1927), 299–361.
19. Denis Talay and Luciano Tubaro, *Expansion of the global error for numerical schemes solving stochastic differential equations*, Stochastic Anal. Appl. **8** (1990), no. 4, 483–509 (1991).
20. Hyek Yoo, *An analytic approach to stochastic partial differential equations and its applications*, ProQuest LLC, Ann Arbor, MI, 1998, Thesis (Ph.D.)—University of Minnesota.
21. ———, *Semi-discretization of stochastic partial differential equations on \mathbf{R}^1 by a finite-difference method*, Math. Comp. **69** (2000), no. 230, 653–666.
22. Moshe Zakai, *On the optimal filtering of diffusion processes*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **11** (1969), 230–243.

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